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The Heterotic Supersymmetric Sigma Model in the Canonical Exterior Formalism

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Abstract Starting from a classical 2D superconformal theory described by the Wess–Zumino–Witten action, the canonical exterior formalism on group manifold for the heterotic supersymmetric sigma model is constructed. The motion equations of the dynamical field and the constraints are found and analyzed from the geometric point of view. It can be seen how the use of the canonical exterior formalism is more adequate and simple because of its manifest covariance in all the steps. The relationship between the form brackets defined in the canonical exterior formalism and the Poisson-brackets is written. Later on, the Dirac-brackets are written by using the second class constraints provided by the canonical exterior formalism. As it can be seen the canonical exterior formalism allows to show how the canonical quantization of the heterotic supersymmetric sigma model is facilitated.

1 Introduction

Recently [1], the supersymmetric extension of the Jackiw–Teitelboim $(1+1)$ linear gravity within the canonical exterior formalism (CEF) on group manifold was constructed. In this context the role of the several fields was analyzed. The constraints and the field equation were found. Finally, this supergravity model was treated in the second order formalism.

From several years ago the interest of the people in studying two-dimensional models has been made evident. Two-dimensional gravity and supergravity models were constructed from different point of view. There is a vast literature on the subject matter and a complete list may not be feasible. In Ref. [1] was only included an incomplete list of works.

The different type of 2D gravity or supergravity models can be briefly enumerated as follows.

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A class of linear gravity theories is based on the Riemann scalar curvature R. The first model of two-dimensional gravity was constructed by Jackiw and Teitelboim (JT) by means of dimensional reduction of the usual Einstein–Hilbert action in $(2+1)$ dimensions [2–6].

Two-dimensional gravitational and vector gauge theories by reduction of $D=3$ topologically massive models were also considered [7].

Later on, by starting from the gauge-theoretic formulation point of view, several works were realized [8–16]. The geometrical structure of the different models obtained in this framework are generally the de Sitter or anti-de Sitter groups (or the corresponding two-dimensional graded ones). All these models have the remarkable property of possessing a topological and gauge invariant formulation. In particular, in Refs. [13, 14] by using non-geometrical fields other type of two-dimensional gravity models were considered. These "string-inspired" models are based on the extended Poincaré group. It is possible to prove that "black-hole" solution appears in this kind of models and so its study becomes interesting from the quantum point of view.

In the last years the aforementioned research engendered much further works [17–21].

For instance, in Refs. [19] and [20], the two-dimensional reduction of the invariant action of the gravitational Chern–Simons model was studied. This was done by means of the Kaluza–Klein like *ansatz*, decomposing the three-dimensional metric into a two-dimensional metric, a $U(1)$ gauge field $A=A_\mu dx^\mu$ and a scalar field ϕ. The dimensional reduction procedure yields a two-dimensional topological theory. In Ref. [19] the main problem was to study local classical solutions, while in Ref. [20] the solutions are extended at global level in order to construct the Carter–Penrose diagrams. It is shown that two types of local classical solutions exist: symmetry breaking and kink solutions. It is interesting to note that the kink make possible an space whose geometry is asymptotically anti-de Sitter. At small distances the scalar curvature is positive and it vanishes at an intermediate point. So, the effect of the kink is analogous to a geometric gravitational force and it can be proved that the resulting two-dimensional action is formally similar to the action of the dilaton model. In Ref. [20], in order to give the discussion to a global level, the action is written by using target space coordinates. As it can be seen, the use of such coordinates brings some advantages from classical as well as quantum point of view [17, 18]. Also, the Bogomolnyi–Prasad–Sommerfield black holes were studied in the framework of the two-dimensional dilaton supergravity [21].

On the other hand, as it is well known the 2D conformal supergravity is the proper framework for the description of superstring theories (see for instance Refs. [22–25] and bibliography quoted therein). This intuitive idea is originated by observing that two is the dimension of the world-sheet (WS) spanned by a one-dimensional object while propagating in an external space-time, named target manifold (M_{target}). The two-dimensional manifolds play an important role because they are responsible for the fundamental geometric structure of superstring theory. Moreover, in order to make local the graded algebra, the two-dimensional vielbein and the two-dimensional gravitino are needed. Of course, in a two-dimensional world, the action reduces to a pure divergence in both cases gravity or supergravity and so, the gravitational field is a non-dynamical one. The gravitational field must be interpreted as a Lagrangian multiplier for the corresponding constraints giving the vanishing condition of the matter fields stress-energy tensor. Consequently, the whole gravitational formalism reduces to a theory of boundary conditions in two-dimension and so, only its topology is the matter of interest. In fact, in the path-integral quantization framework, the two-dimensional different metrics become, after division by the diffeomorphism group, in a discrete sum over the topologies, labeled by a positive integer number g, i.e. the genus of the surface. At fixed topology a multiple integral over a finite dimensional space of complex parameters defines the moduli space M_g, whose coordinates label the conformal classes of the WS. Next, by means of a Wick rotation of the time variable, the superstring WS becomes a Riemann surface which can be treated by using all the results provided by the algebraic geometry.

Historically, the important works given in Refs. [22–24] are devoted to the study of the heterotic sigma model and conformal supergravity in two dimensions. These papers are developed in the context of the Lagrangian formalism in components *via* Noether theorem.

By taking into account this last role of 2D conformal supergravity, and following the idea of Ref. [1] the motivation of the present paper is essentially to study—from a mathematical physics point of view—the supersymmetric sigma model of type II superstring in the framework of the CEF on group manifold. The first advantage is that this formalism is covariant in all its steps. Moreover, because of the direct relation between the form brackets provided by the CEF and the Dirac brackets, the canonical quantization of the heterotic sigma model is facilitated.

The paper is organized as follows: In Sect. 2, the main geometric definitions in 2D superconformal space used in the construction of superconformal field theory are given. The fundamental geometrical quantities of the group manifold $G = M_{\text{target}}$ are written in terms of the left-invariant or right-invariant one-forms containing the Wess–Zumino–Witten field. In Sect. 3, starting from the geometric Lagrangian density which describes the $(1,0)$ heterotic σ model, the CEF on group manifold is constructed. In Sect. 4, the equations of motion are found and their geometrical structure is analyzed. In Sect. 5, the relation between the usual Hamiltonian formalism in components and the CEF is given. Finally, The Dirac-brackets are defined in order to show that starting from the CEF the canonical quantization of the model is facilitated.

2 Definitions and Preliminaries

First, we must consider that every consistent 2D conformal field theory corresponds to a possible string vacuum and it is a suitable starting point for the string perturbation theory. The Green functions of the 2D conformal field theory are then used to construct the string amplitudes. In the geometrical picture, closed string as a one-dimensional loop moving in a smooth target manifold M_{target} was systematically studied (see for instance Ref. [25]). Hence, it is possible to regard as possible string vacua only those consistent conformal theories which are generated by embedding scalar functions $X^\mu(\xi^\alpha)$ from the world-sheet (WS) to that target space ($X^\mu \in M_{\text{target}}, \xi^\alpha \in WS$). In the two-dimensional framework the embedding scalar functions $X^\mu(\xi)$ must be viewed as scalar fields coupled to the 2D gravitational field with metric $g_{\alpha\beta}(\xi)$. The coupling is realized in such a way that the classical action must be invariant under both, diffeomorphisms and Weyl transformations, relating two different 2D conformal metrics. Moreover, in the case of superstrings, the two-dimensional action contains a convenient set of left-handed and right-handed 2D-fermions.

It is clear that a consistent conformal theory implies that the classical conformal theory maintains the classical Virasoro algebra also at the quantum level. This is done by choosing the field content in such a way that after quantization, all the central charges c_i and the coboundaries b_i corresponding to the different fields in the theory, sum up to zero. So, these quantum conformal theories, given by well defined choices of the target space, are suitable string vacua.

As mentioned above the geometric structure underlying heterotic superstring is that of $N = 1$, $D = 2$ conformal supergravity, i.e. the superspace named $(1,0)$. The geometry of this superspace of two bosonic coordinates z and $\bar{z}$ and a single Majorana–Weyl fermionic coordinate θ, is described by a supervielbein (V^+, V^-, ζ) and an S0(1, 1) connection ω. The one-forms (V^+, V^-, ζ) provide a basis for the cotangent space. The one-forms V^+ and V^- are the inner directions and the one-form ζ is the outer direction in the cotangent space. Once the basis (V^+, V^-, ζ) was given, it is possible to write the torsion and the curvature of $(1, 0)$ superspace as follows

$$T^+ = dV^+ + \omega \wedge V^+ = \frac{i}{2}\zeta \wedge \zeta, \tag{1}$$

$$T^- = dV^- - \omega \wedge V^- = 0, \tag{2}$$

$$T^o = d\zeta + \frac{1}{2}\omega \wedge \zeta = \tau V^+ \wedge V^-, \tag{3}$$

$$R = d\omega = \mathcal{R}V^+ \wedge V^- - i\tau\zeta \wedge V^-, \tag{4}$$

where in the right hand side of the above equations are written the correspondent parametrization of torsion and curvature consistent with the corresponding Bianchi identities. In (3), (4) the superfield $\tau(z, \bar{z}, \zeta)$ is the field strength of the two-dimensional gravitino that provides a complete description of the heterotic geometry and $\mathcal{R}$ in (4) is the curvature that equals twice the spinor derivative of τ.

Once (1), (2), (3), (4) are given, the intrinsic covariant derivatives $\mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_o$ remain defined and they satisfy the following algebra

$$[\mathcal{D}_+, \mathcal{D}_-] = \mathcal{R}s + \tau\mathcal{D}_o, \tag{5}$$

$$[\mathcal{D}_o, \mathcal{D}_+] = 0, \tag{6}$$

$$[\mathcal{D}_o, \mathcal{D}_-] = is\tau, \tag{7}$$

$$[\mathcal{D}_o, \mathcal{D}_o] = \frac{i}{2}\mathcal{D}_+, \tag{8}$$

where s is the spin of the field acted on by the derivatives ($s = 0$ for scalars, $s = \frac{1}{2}$ for left-handed fermions and $s = -\frac{1}{2}$ for right-handed fermions).

A classical 2D superconformal theory is described by the Wess–Zumino–Witten action which can be formally written as follows

$$S = \int_{\Sigma_g} d^2\xi \det V(\xi)\mathcal{L}(V^{\pm}, \zeta(\xi), \varphi^i(\xi)), \tag{9}$$

where the integral is defined over the Riemann surface Σ_g which is a 2D real manifold. The one-forms $V^{\pm}$ and ζ are respectively the vielbein and the gravitino fields which are the supergravity background fields, and $\varphi^i(\xi)$ is a convenient set of matter fields.

From several years ago, the geometric action of the $(1, 0)$ σ model was proposed. In order to construct an example of such superconformal theory in the exterior canonical picture, in (9) we take as matter fields $\varphi^i(\xi)$ the components of a superfield $g(z, \bar{z}, \theta)$ which describes the injection

$$g(z, \bar{z}, \theta) : SWS \to G, \tag{10}$$

of the superworld-sheet into a simple group manifold G. This theory is called the Wess–Zumino–Witten model (WZW).

All the geometrical quantities of $M_{\text{target}} = G$ are constructed in terms of the left-invariant or right-invariant one-forms

$$\Omega = g^{-1}dg, \tag{11}$$

$$\bar{\Omega} = dgg^{-1}. \tag{12}$$

So, the Lie algebra-valued one-forms Ω and $\bar{\Omega}$ are decomposed along a basis t_A of the Lie algebra associated to the group manifold G.

$$\Omega = \Omega^A t_A, \tag{13}$$

$$\bar{\Omega} = \bar{\Omega}^A t_A. \tag{14}$$

From the above definition it is obvious that Ω^A and $\bar{\Omega}^A$ satisfy the Maurer–Cartan equations

$$d\Omega^A + \frac{1}{2} f^A_{BC} \Omega^B \wedge \Omega^C = 0, \tag{15}$$

$$d\bar{\Omega}^A - \frac{1}{2} f^A_{BC} \bar{\Omega}^B \wedge \bar{\Omega}^C = 0, \tag{16}$$

for the structure constant f^A_{BC} of the Lie algebra associated to the group manifold G. Since the one-forms Ω^A and $\bar{\Omega}^A$ depend on the superspace coordinates $(z, \bar{z}, \theta)$, they can be written along a complete superspace basis of one-forms

$$\Omega^A = \Omega^A_+ V^+ + \Omega^A_- V^- + \lambda^A \zeta, \tag{17}$$

and similarly for $\bar{\Omega}^A$.

As it was already mentioned, the 2D field theory under review is a particular case of a locally supersymmetric non-linear σ model. The target space metric g^{AB} is in this case the Killing metric $g^{AB} = f^{AMN} f^{BMN}$. Hence it is necessary to introduce a target space spin connection $\omega^{AB} = -\omega^{BA}$ besides of the target space metric. By means of the structure constant f^{ABC} and by looking at the one-form Ω^A as the vielbein of the group manifold G it is possible to introduce a one-parameter family of spin connection defined by

$$\omega^{AB}_{(\alpha)} = \alpha f^{ABC} \Omega^C. \tag{18}$$

The two-forms torsion and curvature associated to the family of connections written in (18) are respectively defined as usual by the expressions

$$T^A_{(\alpha)} = d\Omega^A + \omega^{AB}_{(\alpha)} \wedge \Omega^B = T^{ABC}_{(\alpha)} \Omega^B \wedge \Omega^C, \tag{19}$$

$$R^{AB}_{(\alpha)} = d\omega^{AB}_{(\alpha)} + \omega^{AC}_{(\alpha)} \wedge \omega^{CB}_{(\alpha)} = \mathcal{R}^{ABMN} \Omega^M \wedge \Omega^M, \tag{20}$$

where

$$T^{ABC}_{(\alpha)} = -\left(\alpha + \frac{1}{2}\right) f^{ABC}, \tag{21}$$

$$\mathcal{R}^{ABMN} = -\frac{1}{2}\alpha(1+\alpha) f^{ABC} f^{CMN}. \tag{22}$$

In order to simplify algebraic manipulations, we will work with the most simple geometric action of the $(1,0)$ σ model. The extension to the geometric action of the $(1,1)$ σ model is straightforward [26], and it contains many more terms because of the presence of a second bi-dimensional spinor μ^A.

So, the starting point is to consider the following geometric action

$$\begin{aligned} S(\alpha) = \frac{k}{8\pi}\Bigg\{\int_{M_2}\bigg[&(\Omega^A - \lambda^A\zeta)\wedge(\Omega_+^A V^+ - \Omega_-^A V^-) + 2i\lambda^A\,\nabla_\alpha\lambda^A\wedge V^+ \\ &+ \lambda^A\Omega^A\wedge\zeta - \frac{4i}{3}\left(\frac{1}{2}+\alpha\right) f^{ABC}\lambda^A\lambda^B\lambda^C\zeta\wedge V^+ \\ &+ \Omega_+^A\Omega_-^A V^+\wedge V^-\bigg] + \frac{1}{6}(1+2\alpha)\int_{M_3} f^{ABC}\Omega^A\wedge\Omega^B\wedge\Omega^C\Bigg\}. \end{aligned} \tag{23}$$

The covariant differential of the two-dimensional spinor λ^A is given by

$$\nabla_{(\alpha)}\lambda^A = \mathcal{D}\lambda^A + \omega_{(\alpha)}^{AB}\lambda^B, \tag{24}$$

where is well defined

$$\mathcal{D}\lambda^A \equiv d\lambda^A + \frac{1}{2}\omega\lambda^A. \tag{25}$$

Analogously to (17), the one-form $\mathcal{D}\lambda^A$ is written along a complete superspace basis as follows

$$\mathcal{D}\lambda^A = \mathcal{D}_+\lambda^A V^+ + \mathcal{D}_-\lambda^A V^- + \Gamma^A\zeta, \tag{26}$$

where the outer component Γ^A is given by

$$\Gamma^A = -\frac{i}{2}\Omega_+^A + \frac{1}{2} f^{ABC}\lambda^B\lambda^C. \tag{27}$$

In the geometric action (23) the dynamical variables are given by

(a) The supergravity background one-form fields V^+, V^-, ζ.
(b) The WZW field g contained in the one-form field Ω^A and its superpartner λ^A.
(c) The auxiliary 0-forms fields Ω_+^A, Ω_-^A which play a double role: (i) enforces the rheonomic parametrization, and (ii) the field equation yields $\Omega_\pm^A = \mathrm{tr}(g^{-1}\partial_\pm g t^A)$.

The above action for the WZW model is an example of a locally supersymmetric two-dimensional heterotic σ model.

3 Canonical Exterior Formalism on Group Manifold for the Heterotic Supersymmetric Sigma Model

The CEF was constructed and applied to different models of gravity and supergravity in diverse dimensions, as well as their coupling to matter supermultiplets and to the Yang–Mills field [27–36]. The general response is that this formalism permits to find and study constraints, equation of motion and all the dynamical properties of such systems in a more simple way that following the usual Lagrangian method. As it was already commented, the CEF is covariant in all its steps because it is constructed by using only operation of the exterior algebra. As it is obvious, in all the above papers the gravity or supergravity fields are dynamics ones.

In the present paper the idea is to work by first time with the CEF applied to the description of the heterotic supersymmetric sigma model in which the supergravity field is a non-dynamical one.

In (23) there are three critical values of α: $\alpha = -\frac{1}{2}$; $\alpha = 0$ and $\alpha = -1$. We will consider the $\alpha = -\frac{1}{2}$ case, which corresponds to choose a metric connection for which the torsion equation (19) is zero. Therefore, in such case the Lagrangian density is written as follows:

$$\begin{aligned}\mathcal{L} = \frac{k}{8\pi}[(\Omega^A - \lambda^A \zeta) \wedge (\Omega_+^A V^+ - \Omega_-^A V^-) + 2i\lambda^A \nabla \lambda^A \wedge V^+ \\ + \lambda^A \Omega^A \wedge \zeta + \Omega_+^A \Omega_-^A V^+ \wedge V^-].\end{aligned} \tag{28}$$

In the Lagrangian density (28) the auxiliary two 0-forms fields Ω_+^A, Ω_-^A are non-geometrical objects and are introduced with the purpose of obtaining rheonomic equations of motion, i.e. equations compatible with the Bianchi identities as it is required by the group manifold approach [27, 28].

In order to obtain the equation of motion, instead of the WZW field g contained in the one-form field Ω^A, we can use as dynamical variable the tangent variation i.e.

$$\delta y = g^{-1}\delta g = \delta y^A t_A, \tag{29}$$

which is related to the variation of the one-form Ω^A by

$$\delta\Omega^A = d\delta y^A + f^{ABC}\Omega^B dy^C. \tag{30}$$

The use of the 0-forms variables (y^A , λ^A) allows to obtain, for both variables, equations of motion having the same structure. So, having in mind (29) and (30), and making in (28) the change of variables, apart from a total exterior derivative the Lagrangian density writes

$$\begin{aligned}\mathcal{L} = {} & -\lambda^A \zeta \wedge \Pi^A - y^A \wedge d\Pi^A \\ & + f^{ABC}\Omega^B y^C \wedge \Pi^A - 2id\lambda^A\lambda^A \wedge V^+ \\ & + 2if^{ABC} d\lambda^A \lambda^B y^C \wedge V^+ \\ & + if^{ABC}\lambda^A\lambda^B y^C dV^+ \\ & - if^{ABC} f^{CDE}\lambda^A\lambda^B\Omega^D y^E \wedge V^+ \\ & - d\lambda^A y^A \wedge \zeta + d\zeta\lambda^A y^A \\ & + f^{ABC}\lambda^A\Omega^B y^C \wedge \zeta + \Omega_+^A\Omega_-^A V^+ \wedge V^-,\end{aligned} \tag{31}$$

where was defined

$$\Pi^A = \Omega_+^A V^+ - \Omega_-^A V^-. \tag{32}$$

Therefore, (31) is our starting point in order to construct the first-order CEF.

By following Ref. [34], we define the canonical conjugate momenta to each one of the dynamical field variables $\mu^\Sigma = (y^A, \lambda^A, V^+, V^-, \zeta, \Omega_+^A, \Omega_-^A)$ for the compound index Σ.

By means of the functional variation of the Lagrangian with respect to the “velocities” $d\mu^{\Sigma}$, i.e.: $\pi_{\Sigma} = \delta\mathcal{L}/\delta d\mu^{\Sigma}$, the canonical conjugate momenta remain defined as follows:

(i) The momenta associated with the 0-forms y^A, λ^A respectively read

$$P^A = \frac{\delta\mathcal{L}}{\delta(dy^A)} = 0 \tag{33}$$

bosonic one-form, and

$$Q^A = \frac{\delta\mathcal{L}}{\delta(d\lambda^A)} = -2i\lambda^A V^+ + 2if^{ABC}\lambda^B y^C V^+ - y^A\zeta \tag{34}$$

fermionic one form.

(ii) The momenta associated with the supergravity background one-form fields V^+, V^-, ζ are:
the bosonic 0-form

$$\pi_+ = \frac{\delta\mathcal{L}}{\delta(dV^+)} = if^{ABC}\lambda^A\lambda^B y^C - y^A\Omega_+^A, \tag{35}$$

the bosonic 0-form

$$\pi_- = \frac{\delta\mathcal{L}}{\delta(dV^-)} = y^A\Omega_-^A, \tag{36}$$

and the fermionic 0-form

$$\pi_\zeta = \frac{\delta\mathcal{L}}{\delta(d\zeta)} = \lambda^A y^A. \tag{37}$$

(iii) The momenta associated with the 0-forms Ω_+^A, Ω_-^A are respectively the following bosonic one-form

$$\mathcal{P}_+^A = \frac{\delta\mathcal{L}}{\delta(d\Omega_+^A)} = -y^A V^+, \tag{38}$$

and

$$\mathcal{P}_-^A = \frac{\delta\mathcal{L}}{\delta(d\Omega_-^A)} = y^A V^- \tag{39}$$

bosonic one-form.

In the CEF, it is necessary to define a suitable operation involving forms, capable of replacing the role of the classical Poisson brackets. Therefore, the graded form-brackets operation between pairs of canonical variables is defined and it is given by

$$(\mu^{\Sigma}, \pi_{\Lambda}) = (-1)^{a+1+|A|}\delta_{\Lambda}^{\Sigma}, \tag{40}$$

where a and $|A|$ are respectively the degree and the Fermi grading of the form μ^{Σ}. The remaining form-brackets properties for generic superforms were written in (2.2) of Ref. [34].

In the present case, the form-brackets between pairs of canonical variables write

$$(y^A, P_B) = (P_B, y^A) = -\delta_B^A, \tag{41}$$

$$(\lambda^A, Q_B) = -(Q_B, \lambda^A) = \delta_B^A, \tag{42}$$

$$(V^+, \pi_+) = (\pi_+, V^+) = 1, \tag{43}$$

$$(V^-, \pi_-) = (\pi_-, V^-) = 1, \tag{44}$$

$$(\zeta, \pi_\zeta) = -(\pi_\zeta, \zeta) = -1, \tag{45}$$

$$(\Omega_+^A, \mathcal{P}_+^B) = (\mathcal{P}_+^B, \Omega_+^A) = -\delta^{AB}, \tag{46}$$

$$(\Omega_-^A, \mathcal{P}_-^B) = (\mathcal{P}_-^B, \Omega_-^A) = -\delta^{AB}. \tag{47}$$

The set of momenta (33–39) defines the following primary constraints

$$\Phi^A = P^A \approx 0, \tag{48}$$

$$\Psi^A = Q^A + 2i\lambda^A V^+ - 2if^{ABC}\lambda^B y^C V^+ + y^A \zeta \approx 0, \tag{49}$$

$$\varphi_+ = \pi_+ - if^{ABC}\lambda^A\lambda^B y^C + y^A \Omega_+^A \approx 0, \tag{50}$$

$$\varphi_- = \pi_- - y^A \Omega_-^A \approx 0, \tag{51}$$

$$\varphi_\zeta = \pi_\zeta - \lambda^A y^A \approx 0, \tag{52}$$

$$\Theta_+^A = \mathcal{P}_+^A + y^A V^+ \approx 0, \tag{53}$$

$$\Theta_-^A = \mathcal{P}_-^A - y^A V^- \approx 0. \tag{54}$$

By considering the definition and properties of the graded form-brackets written in (2.2) of Ref. [34], it is possible to compute the form-brackets $(\Phi^\Sigma, \Phi^\Lambda)$ for pairs of constraints. It is straightforward to prove that all the primary constraints (48–54) are second-class ones, that is

$$(\Phi^\Sigma, \Phi^\Lambda) \neq 0. \tag{55}$$

In the CEF, the conserved first-class dynamical quantity describing the dynamics of the system is the extended Hamiltonian H_T, and it is the bosonic two-form defined by (see Ref. [34])

$$H_T = H_{\text{can}} + \Lambda^\Sigma \wedge \Phi_\Sigma, \tag{56}$$

where the Lagrange multipliers Λ^Σ can be unambiguously determined. When the fundamental equation of motion in the CEF is taken into account, it is possible to write the Hamiltonian equations for pairs of canonical variables,

$$d\mu^\Sigma = (\mu^\Sigma, H_T), \tag{57}$$

$$d\pi^\Sigma = (\pi^\Sigma, H_T). \tag{58}$$

From (57) and by using (56) the following general result is obtained

$$\Lambda^\Sigma = d\mu^\Sigma. \tag{59}$$

In (56) the canonical Hamiltonian $H_{\mathrm{can}} = d\mu^{\Sigma} \wedge \pi_{\Sigma} - \mathcal{L}$ is given by

$$\begin{aligned} H_{\mathrm{can}} = {} & dy^A \wedge P_A + d\lambda^A \wedge Q_A + dV^+ \wedge \pi_+ + dV^- \wedge \pi_- \\ & + d\zeta \wedge \pi_\zeta + d\Omega_+^A \wedge \mathcal{P}_+^A + d\Omega_-^A \wedge \mathcal{P}_-^A - \mathcal{L} \end{aligned} \tag{60}$$

which after using (31) for the Lagrangian it results

$$\begin{aligned} H_{\mathrm{can}} = {} & \lambda^A \zeta \wedge (\Omega_+^A V^+ - \Omega_-^A V^-) \\ & - f^{ABC} \Omega^B y^C \wedge (\Omega_+^A V^+ - \Omega_-^A V^-) \\ & + i f^{ABC} f^{CDE} \lambda^A \lambda^B \Omega^D y^E \wedge V^+ \\ & - f^{ABC} \lambda^A \Omega^B y^C \wedge \zeta \\ & - \Omega_+^A \Omega_-^A V^+ \wedge V^-. \end{aligned} \tag{61}$$

4 Equations of Motion in the Canonical Exterior Formalism

The field equations of motion in the CEF are given by the consistency conditions on the primary constraints, i.e.

$$d\Phi^{\Sigma} = (\Phi^{\Sigma}, H_T) \approx 0. \tag{62}$$

As it was commented above the vielbein and the gravitino are not dynamical fields in 2D, therefore the motion equation for the supergravity background fields V^+, V^- and ζ will be not considered. The supergravity background fields play the role of Lagrange multipliers associated to the primary constraints of the theory, that is the superstress-energy tensor and the supercurrent. In fact, the superstress-energy tensor and the supercurrent one-forms are respectively defined by making the variation of the action (23) with respect to the supervielbein (V^+, V^-, ζ). As it is known in the classical theory these quantities are weakly zero ones. From the quantum point of view they are used to construct the BRST-charge.

About the variables Ω_+^A and Ω_-^A we remember that they are introduced to enforce the rheonomic parametrization.

Therefore, the main equations are those for the fields y^A and λ^A which respectively read

$$\begin{aligned} d\Phi^M = {} & (\Phi^M H_T) \\ = {} & (P^M, H_{\mathrm{can}}) + \Lambda^B \wedge (\Phi^M, \Phi_B) + \Sigma^B \wedge (\Phi^M, \Psi_B) \\ & + \Lambda_+ \wedge (\Phi^M, \varphi_+) + \Lambda_- \wedge (\Phi^M, \varphi_-) + \Sigma_\zeta \wedge (\Phi^M, \varphi_\zeta) \\ & + \Delta_+^A \wedge (\Phi^M, \Theta_+^A) + \Delta_-^A \wedge (\Phi^M, \Theta_-^A) \\ & + \text{weakly zero terms} = 0, \end{aligned} \tag{63}$$

$$\begin{aligned} d\Psi^M = {} & (\Psi^M H_T) \\ = {} & (Q^M, H_{\mathrm{can}}) + dy^B \wedge (\Psi^M, \Phi_B) - d\lambda^B \wedge (\Psi^M, \Psi_B) \\ & + dV^+ \wedge (\Psi^M, \varphi_+) + dV^- \wedge (\Psi^M, \varphi_-) + d\zeta \wedge (\Psi^M, \varphi_\zeta) \\ & + d\Omega_+^A \wedge (\Psi^M, \Theta_+^A) + d\Omega_-^A \wedge (\Psi^M, \Theta_-^A) \\ & + \text{weakly zero terms} = 0. \end{aligned} \tag{64}$$

Now, the following explicit expressions for the form-brackets between constraints, must be considered

$$(\Phi^A, \Phi^B) = 0, \tag{65}$$

$$(\Phi^A, \Psi^B) = 2if^{ABC}\lambda^C V^+ - \zeta\delta^{AB}, \tag{66}$$

$$(\Phi^A, \varphi_+) = if^{ABC}\lambda^B\lambda^C - \Omega^A_+, \tag{67}$$

$$(\Phi^A, \varphi_-) = \Omega^A_-, \tag{68}$$

$$(\Phi^A, \varphi_\zeta) = \lambda^A, \tag{69}$$

$$(\Psi^A, \Psi^B) = -4iV^+\delta^{AB}, \tag{70}$$

$$(\Psi^A, \varphi_+) = 2i\lambda^A, \tag{71}$$

$$(\Psi^A, \varphi_-) = 0, \tag{72}$$

$$(\Psi^A, \varphi_\zeta) = 0, \tag{73}$$

$$(\Phi^A, \Theta^B_+) = -\delta^{AB}V^+, \tag{74}$$

$$(\Phi^A, \Theta^B_-) = \delta^{AB}V^-, \tag{75}$$

$$(\Psi^A, \Theta^B_+) = (\Psi^A, \Theta^B_-) = 0, \tag{76}$$

$$(\varphi_+, \Theta^A_+) = (\varphi_+, \Theta^A_-) = 0, \tag{77}$$

where the form brackets between constraints involved in (63) and (64) were only written.

By replacing the above expressions for the form-brackets between constraints, in (63) and (64), they respectively read

$$\begin{aligned} d\Phi^M = -\Big(& d\Pi^A - f^{ABC}\Omega^B \wedge \Pi^C + \mathcal{D}\lambda^A \wedge \zeta + \lambda^A T^o \\ & + f^{ABC}\Omega^B\lambda^C \wedge \zeta - 2if^{ABC}\lambda^B\mathcal{D}\lambda^C \wedge V^+ + \frac{1}{2}f^{ABC}\lambda^B\lambda^C\zeta \wedge \zeta \\ & - if^{ABC}f^{CDE}\Omega^B\lambda^D\lambda^E \wedge V^+\Big) + \text{weakly zero terms} = 0, \end{aligned} \tag{78}$$

$$\begin{aligned} d\Psi^M = & -(-4i\nabla\lambda^A \wedge V^+ + \zeta \wedge \Pi^A + \zeta \wedge \Omega^A + \lambda^A\zeta \wedge \zeta) \\ & + \text{weakly zero terms} = 0. \end{aligned} \tag{79}$$

Both equations (78) and (79) defined over the heterotic superspace are two-forms. Having the same structure they can be decomposed into four independent sectors corresponding to the inner-inner direction $V^+ \wedge V^-$ the inter-outer directions $V^+ \wedge \zeta$ and $V^- \wedge \zeta$ and the outer-outer direction $\zeta \wedge \zeta$.

The first step is to consider the Maurer–Cartan two-form equation (15) and the one-forms defined in (17), (26) and (27) decomposed along the supergravity background one-form fields (V^+, V^-, ζ).

By straightforward calculation it can be shown:

(i) Considering (78) it can be seen that the coefficients of the components $V^+ \wedge \zeta$, $V^- \wedge \zeta$ and $\zeta \wedge \zeta$ cancel automatically when the rheonomic parametrization (see (17), (26) and (27)) is introduced. On the other hand, the cancellation of the component $V^+ \wedge V^-$ gives rise to the following condition

$$\mathcal{D}_- \Omega_+^A + \mathcal{D}_+ \Omega_-^A - \tau \lambda^A - 2if^{ABC} \lambda^B \mathcal{D}_- \lambda^C - if^{ABC} f^{CDE} \Omega_-^B \lambda^D \lambda^E = 0. \tag{80}$$

(ii) Analogously, by considering (79) it can be seen that the coefficients of the components $V^+ \wedge \zeta$, $V^- \wedge \zeta$ and $\zeta \wedge \zeta$ cancel automatically, while the cancellation of the component $V^+ \wedge V^-$ gives rise to the following condition

$$\mathcal{D}_- \lambda^A - \frac{1}{2} f^{ABC} \lambda^B \Omega_-^C = 0. \tag{81}$$

Now, considering the different projections for the Maurer–Cartan equation (15), the following conditions are found:

(iii) The coefficient cancellation of the components $V^+ \wedge \zeta$ and $V^- \wedge \zeta$ implicates respectively the following conditions

$$\mathcal{D}_o \Omega_+^A - \mathcal{D}_+ \lambda^A - f^{ABC} \Omega_+^B \lambda^C = 0, \tag{82}$$

$$\mathcal{D}_o \Omega_-^A - \mathcal{D}_- \lambda^A - f^{ABC} \Omega_-^B \lambda^C = 0, \tag{83}$$

and the cancellation of the coefficient of $V^+ \wedge V^-$ gives rise to the Bianchi identity, i.e.

$$\mathcal{D}_+ \Omega_-^A - \mathcal{D}_- \Omega_+^A - \tau \lambda^A + f^{ABC} \Omega_+^B \Omega_-^C = 0. \tag{84}$$

The coefficient of $\zeta \wedge \zeta$ cancel automatically.

Therefore, the conclusion is that the motion field equations (78) and (79) for the fields y^A and λ^A, are reduced to the two differential equations (80) and (81), and the remaining conditions are all geometrical ones.

5 Usual Hamiltonian Formalism Versus Canonical Exterior Formalism

From the above construction it can be seen that the CEF is covariant in all the steps. But, as it was commented in detail in Ref. [34] the CEF is not a proper Hamiltonian formalism because the extended Hamiltonian H_T defined in (56) is not a true generator of time evolutions. The form-brackets do not contain the same information as the Poisson brackets. Really, the Poisson-brackets contain more information than the form-brackets defined in the CEF. In fact, the CEF can be related with the Hamiltonian formalism in components, and so the form-brackets are related to the Poisson brackets but not in a trivial way [28, 34]. The integral relationship which relates the form-brackets (A, B) to the Poisson brackets between forms $[A(x), B(y)]$ is given by

$$(-1)^{a+1} \int_\Sigma \alpha \wedge (A, B) \wedge \beta = \iint_{\Sigma \times \Sigma} \alpha(x) \wedge [A(x), B(y)] \wedge \beta(y), \tag{85}$$

where a is the degree of the form A and α, β are text forms.

On the other hand, it is well known that the second order formalism is necessary when the model is considered from the quantum point of view. In fact, it is in the second order formalism where the dynamical degrees of freedom are separated from those of gauge degrees of freedom.

In Ref. [1] Sect. 4, the second order formalism by solving the torsion field equation in two-dimensional supergravity models was studied in detail. In this paper we only reproduce a few useful concepts.

When the spacetime decomposition is considered and a privileged time direction is chosen in the manifold M^2, the manifest covariance is lost. Usually, the time variable is chosen so that the one-form dx^0 can be detached. More precisely, we consider fields and forms defined on a spacelike $x^0 = t = t^0$ one-dimensional "surface" Σ, by defining the injection map $\chi : \Sigma \to M^2$. Thus, the associated pullback χ^* acts on any form by setting $t = t^0$ and $dt^0 = 0$.

Once the space-time decomposition is done and the surface Σ remains defined, the ordinary Poisson brackets are obtained by expanding the forms $A(x)$ and $B(y)$, given in (85), in the holonomic bases dx^i, dy^j. Then, the ordinary Poisson brackets between fields and momenta components can be used.

All the quantities provided by the CEF, i.e. the total Hamiltonian, the constraints and the field equations must be projected on the "surface" Σ. Once the canonical conjugate momenta π_A are written in terms of the spatial components dx^i of the holonomic basis, the Poisson brackets between pairs of canonical variables remain defined as usual.

The final form of the Hamiltonian as the generator of time evolutions in the canonical component formalism is obtained by taking into account the metricity condition in one and two dimensions (see (50–54), Sect. 4 of Ref. [1]).

Another question to take into account is that the CEF plays, with respect to the first order canonical component formalism, an analogous role to that played by the first order canonical component formalism with respect to the second order formalism. Therefore, we will consider that all the primary constraints in the CEF remain at least weakly zero in the canonical component formalism (see, for instance Refs. [28, 34]). Consequently, we assume that the restrictions to Σ of the constraints (48), (49), (52–54) are strongly equal to zero. For the remaining constraints (50), (51) $\varphi_\pm$ the restriction to Σ is maintained as a weakly zero quantity, i.e.:

$$\chi^* \varphi_\pm \approx 0. \tag{86}$$

The bosonic 2-form (56) provides by the CEF can be written as follows

$$\int H_T = \int dx^0 \tilde{\mathcal{H}}, \tag{87}$$

where the time variable is chosen so that the 1-form dx^0 can be detached. The remaining bosonic one-form integrated in one dimension is the proper Hamiltonian generator of time evolutions and it turns out to be of the form

$$\tilde{\mathcal{H}} = \int dx \left(\frac{1}{2} \omega_0 \mathcal{H} + V_{a0} \mathcal{H}^a + \bar{\xi}_{0\alpha} \mathcal{H}^\alpha \right). \tag{88}$$

Finally, it can be proven that the constraints $\mathcal{H}$, $\mathcal{H}^a$ and $\mathcal{H}_\zeta$ are the first-class constraints closing the following constraint superalgebra

$$[\mathcal{H}_A(x), \mathcal{H}_B(y)] = \Lambda^C_{AB} \mathcal{H}_C(x) \delta(x - y), \tag{89}$$

where $\Lambda^C_{AB} = R^C_{AB} - C^C_{AB}$ are the *structure functions* for curvatures R^C_{AB} and structure constant C^C_{AB} of the graded Lie algebra. In particular it is easy to see that the antisymmetric weakly zero quantity $\mathcal{H}$ that appears in (88) is the generator of local Lorentz rotations, that in context of the CEF naturally appears when the space-time decomposition is carried out. Contrarily, starting from the component Hamiltonian formalism, the generator of local Lorentz rotations must be introduced *ad hoc* by demanding the closure of the constraint algebra.

By following the same steps as those given in Sect. 4 of Ref. [1], it is straightforward to explicitly write the first class constraints that verify (89).

Before concluding this section a further consideration about the exterior canonical formalism must be done: as it was said, all the primary constraints provided by the CEF are second-class ones, and so they are not related with the gauge symmetry of the model. Moreover, the possibility of using different Lagrangian densities means that there is not a unique set of canonical conjugate momenta and consequently there is not a unique set of primary constraints in the CEF. On the other hand, in the second-order formalism the second-class constraints must be eliminated. This is done by defining the Dirac brackets from the Poisson brackets. As it is well known the Dirac brackets $[F, G]^D$ for generic functional F and G are obtained from the set of second-class constraints Ψ_A by means of the definition

$$[F, G]^D = [F, G] - [F, \Psi_A] C^{AB} [\Psi_B, G], \tag{90}$$

where $C^{AB}[\Psi_B, \Psi_C] = \delta^A_C$ for the compound indices A, B, C. To compute the Dirac brackets (86) we must consider the restriction to Σ of all the second-class constraints (48–54).

As it is known, the main properties of the Dirac brackets are:

(i) If one of the function F or G is first class, then

$$[F, G]^D \approx [F, G]. \tag{91}$$

In particular, for the Hamiltonian $\mathcal{H}$ holds

$$[F, \mathcal{H}]^D \approx [F, \mathcal{H}]. \tag{92}$$

This means that the same equations of motion are obtained by using the Poisson or the Dirac brackets. Thus, the rate of change in time of any functional F of the canonical variables is also given by

$$\dot{F} = [F, \mathcal{H}]^D. \tag{93}$$

(ii) For any functional F of the canonical variables it is

$$[\Psi_A, \mathcal{H}]^D = 0. \tag{94}$$

Therefore, we can set $\Psi_A = 0$ either before or after evaluating the Dirac brackets.

Once the Dirac brackets are evaluated from (86), the transition to quantum theory is realized as usual in a canonical formalism by replacing classical fields by quantum field operators acting on some Hilbert space. Consequently, the canonical Dirac brackets are replaced by quantum commutators between field operators.

Finally, let us a few words about the well known canonical quantization procedure ... in the usual literature.

We consider as an example the WZW field $g(z, \bar{z})$ defined in (10) for the bosonic case. It can be shown [25], that Ω^A_+ becomes a conserved chiral current i.e., $\mathcal{D}_- \Omega^A_+ = 0$. This last equation implies $\mathcal{D}_+\bar{\Omega}_-^A = 0$. Therefore, out of an on-shell WZW field $g(z,\bar{z})$ two analytic currents can be constructed

$$J^A(z) = -i\frac{k}{2}\Omega_+^A = i\frac{k}{2}\Omega_z^A = -i\frac{k}{2}\,\mathrm{tr}(g^{-1}\partial_z g t^A), \tag{95}$$

$$\bar{J}^A(\bar{z}) = -i\frac{k}{2}\bar{\Omega}_-^A = i\frac{k}{2}\bar{\Omega}_{\bar{z}}^A = -i\frac{k}{2}\,\mathrm{tr}(\partial_{\bar{z}} g g^{-1} t^A). \tag{96}$$

Now, as it is known from the current literature, the Dirac brackets of the dynamical variables $J^A(z)$ and $\bar{J}^A(\bar{z})$ write as follows

$$[J^A(z), J^B(w)]^D = i\frac{f^{ABC}J^C(w)}{(z-w)} + \frac{k}{2}\frac{\delta^{AB}}{(z-w)^2}, \tag{97}$$

$$[J^A(z), \bar{J}^B(\bar{w})]^D = 0, \tag{98}$$

$$[\bar{J}^A(\bar{z}), \bar{J}^B(\bar{w})]^D = i\frac{f^{ABC}J^C(w)}{(\bar{z}-\bar{w})} + \frac{k}{2}\frac{\delta^{AB}}{(\bar{z}-\bar{w})^2}. \tag{99}$$

After performing a Laurent series expansion of the two fields $J^A(z)$ and $\bar{J}^A(\bar{z})$ and replacing the canonical Dirac brackets by the quantum commutators between field operators the canonical quantization is realized.

6 Conclusions

We conclude that due to its intrinsic geometrical language, the CEF can be used as an interesting formal resource to understand the structure of the supergravity field theories in diverse dimensions, as well as the heterotic supersymmetric sigma model describing type II superstring.

The first remark is that the CEF is not a proper canonical formalism because it does not propagate data defined on an initial surface as it is required by a standard mechanical system. However, as it can be seen from the above construction, that the CEF is a powerful method at classical level. Due to the covariance of the CEF in all its steps this formalism allows to find the equations of motion and the constraints in a very simple way without introducing complicate algebraic manipulations.

Since all the primary constraints coming from the CEF are second-class ones, the Dirac brackets are easily defined by projecting these constraints on the surface Σ.

The relation between the CEF and the usual first-order canonical formalism written in components, was also given. This relation was done by means of a nontrivial integral relationship between the form-brackets and the usual Poisson brackets.

In order to go over the second-order formalism, the space-time decomposition in M^2 was performed, losing the explicit covariance of all the equations. Once this is done, the Hamiltonian system is treated as usual according to the Dirac prescriptions. From the total Hamiltonian coming from the CEF, it is evaluated the proper Hamiltonian (88) as the generator of time evolution. As it was shown, the primary constraint obtained in the CEF also plays an important role in the construction of the proper Hamiltonian (88). Precisely, it is given in terms of the first-class constraints which close the constraint algebra. Therefore, all the Hamiltonian gauge symmetries remain determined and the apparent gauge degrees of freedom can be unambiguously removed leaving only the physical ones. When the model is considered from the quantum point of view this last step is necessary.

Finally, the CEF can be used for describing the heterotic σ model on a general target space M_{target} of dimension $D = d_{\mathrm{Mink}} + d_{\mathrm{compact}}$.

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